Backlund transformations for the isospectral and non-isospectral MKdV hierarchies

1990 J. Phys. A: Math. Gen. 23 2867

(http://iopscience.iop.org/0305-4470/23/13/024)

J. Phys. A: Math. Gen. 23 (1990) 2867-2877. Printed in the UK

Bäcklund transformations for the isospectral and non-isospectral MKdV hierarchies

Tian Chou and Zhang Youjin

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, People's Republic of China

Received 15 September 1989

Abstract. By transforming the usual Lax pairs of the isospectral and non-isospectral MKdV hierarchies into Lax pairs in Riccati form, a unified explicit form of Bäcklund transformations and superposition formulae for these hierarchies of equations can be obtained.

1. Introduction

As a powerful means in the construction of solutions for non-linear evolution equations, the Bäcklund transformation has become a very important subject in the study of nonlinear evolution equations [1-6]. During the past few years, it has been noticed that by using the Darboux matrix method, a unified explicit form of Bäcklund transformations can be obtained for some hierarchies of isospectral equations, such as isospectral KdV, MKdV, sine-Gordon and the AKNS hierarchy [7-11]. The approach to the study consists of constructing the Darboux matrix first, and then proving the gauge equivalence of the related Lax pairs. However, demonstrating the t part is quite difficult in this approach, and it is also hard to employ this method to study hierarchies of non-isospectral evolution equations.

In the present paper, we firstly convert the usual Lax pairs for the isospectral and non-isospectral MKdV hierarchies into Lax pairs in Riccati form, then by using an obvious invariability for the t part of the new Lax pair, we obtain concisely a unified explicit form of Bäcklund transformations for the isospectral and non-isospectral MKdV hierarchies, and we also obtain a superposition law for these hierarchies of equations. The advantage of our approach is that it not only enables us to get Bäcklund transformations for the isospectral and non-isospectral hierarchies in a unified way, but also makes the procedure much simpler and clearer than in the previous method. Furthermore, our approach can also be employed to study other hierarchies of equations, such as the isospectral and non-isospectral KdV and AKNS hierarchies [12].

For clarity, we first consider the isospectral MKdV hierarchy in section 2, and then consider the non-isospectral case in section 3.

2. Bäcklund transformations for the isospectral MKdV hierarchy

In this paper we always assume that $q(x, t)$ is a smooth function of x and t, and q and its derivatives to any other with respect to x tend to zero rapidly when $x \to -\infty$.

Consider the isospectral MKdv hierarchy

$$q_t = K_n = \Psi^n q_x \qquad n = 0, 1, 2, \ldots \tag{2.1}$$

where

$$\Psi = \Psi(q) = \mathrm{D}^2 + 4q^2 + 4q_x \mathrm{D}^{-1} q = \mathrm{D}(\mathrm{D} + 4q\mathrm{D}^{-1} q)$$

and

$$\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x} \qquad \mathrm{D}^{-1} = \int_{-x}^{x} \mathrm{d}x' \qquad \Psi^0 = I.$$

Equation (2.1) has the following Lax pair [13]:

$$V_x = MV \qquad V_t = N_n V \tag{2.2}$$

where

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad M = \begin{pmatrix} \eta & q \\ -q & -\eta \end{pmatrix} \qquad N_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix}$$

η is the spectral parameter with $\eta_t = 0$ (due to this we call the hierarchy of equations (2.1) the isospectral hierarchy, and the case $\eta_t \neq 0$ corresponds to non-isospectral hierarchy), and A_n, B_n, C_n are defined by

$$B_n + C_n = \sum_{l=1}^{n} 4^l \eta^{2l-1} \Psi^{n-l} \mathrm{D} q \qquad B_0 + C_0 = 0 \qquad n = 1, 2, \ldots \tag{2.3}$$

$$\begin{aligned} B_n - C_n &= \sum_{l=0}^{n-1} 2(4\eta^2)^l (\mathrm{D} + 4q\mathrm{D}^{-1} q)\Psi^{n-l-1} \mathrm{D} q + 2(4\eta^2)^n q \\ &= \sum_{l=0}^{n} 2(4\eta^2)^l [(\mathrm{D} + 4q\mathrm{D}^{-1} q)\mathrm{D}]^{n-l} q \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{2.4}$$

$$A_n = \mathrm{D}^{-1} q(B_n + C_n) + (4\eta^2)^n \eta \qquad n = 0, 1, 2, \ldots. \tag{2.5}$$

Let $\Gamma = v_2/v_1$, then from (2.2) we have

$$\Gamma_x = -q(1+\Gamma^2) - 2\eta\Gamma \tag{2.6a}$$

$$\Gamma_t = C_n - 2A_n\Gamma - B_n\Gamma^2 = \tfrac{1}{2}(C_n - B_n)(1+\Gamma^2) + \tfrac{1}{2}(C_n + B_n)(1-\Gamma^2) - 2A_n\Gamma. \tag{2.6b}$$

Define $\varphi = \tan^{-1}\Gamma$, then (2.6) can be written as

$$\varphi_x = -q - \eta \sin 2\varphi \tag{2.7a}$$

$$\varphi_t = \tfrac{1}{2}(C_n - B_n) + \tfrac{1}{2}(C_n + B_n)\cos 2\varphi - A_n \sin 2\varphi. \tag{2.7b}$$

From the boundary conditions of q we can assume that φ and its derivatives to any order with respect to x tend to zero rapidly when $x \to -\infty$. (If φ tends to infinity we can assume $\Gamma = v_1/v_2$.) Now we define

$$\tilde{\Psi}(\varphi, \eta) = \mathrm{D}^2 + 4\varphi_x^2 - 4\varphi_x \mathrm{D}^{-1}\varphi_{xx} + 4\eta^2 \sin^2 2\varphi + 8\eta^2 \varphi_x \mathrm{D}^{-1} \sin 2\varphi \cos 2\varphi$$

$$T(\varphi, \eta) = \mathrm{D} + 2\eta \cos 2\varphi.$$

It is easy to prove that

$$\Psi(q)T(\varphi, \eta) = T(\varphi, \eta)\tilde{\Psi}(\varphi, \eta). \tag{2.8}$$

By using the above formula we have the following lemma.

Lemma 2.1. If $q(x, t)$ and φ are related by (2.7*a*), then

$$\tfrac{1}{2}(C_n - B_n) + \tfrac{1}{2}(C_n + B_n)\cos 2\varphi - A_n \sin 2\varphi = \tilde{\Psi}(\varphi, \eta)^n \varphi_x.$$

Proof. From (2.3)-(2.5) we have

$$
\begin{aligned}
&\tfrac{1}{2}(C_n - B_n) + \tfrac{1}{2}(C_n + B_n)\cos 2\varphi - A_n \sin 2\varphi \\
&\quad = -\sum_{l=0}^{n-1} (4\eta^2)^l (\mathrm{D} + 4q\mathrm{D}^{-1}q)\Psi^{n-l-1}\mathrm{D}q - (4\eta^2)^n q \\
&\qquad + \cos 2\varphi \sum_{l=1}^{n} 2^{2l-1}\eta^{2l-1}\Psi^{n-l}q_x \\
&\qquad - 2\sin 2\varphi \sum_{l=1}^{n} 2^{2l-1}\eta^{2l-1}\mathrm{D}^{-1}q\Psi^{n-l}q_x - (4\eta^2)^n\eta \sin 2\varphi \\
&\quad = \sum_{l=0}^{n-1} (4\eta^2)^l (\mathrm{D} + 4q\mathrm{D}^{-1}q)\Psi^{n-l-1}T\varphi_x + (4\eta^2)^n(\varphi_x + \eta \sin 2\varphi) \\
&\qquad - \cos 2\varphi \sum_{l=1}^{n} 2^{2l-1}\eta^{2l-1}\Psi^{n-l}T\varphi_x \\
&\qquad + \sin 2\varphi \sum_{l=1}^{n} 2^{2l}\eta^{2l-1}\mathrm{D}^{-1}q\Psi^{n-l}T\varphi_x - (4\eta^2)^n\eta \sin 2\varphi \\
&\quad = \sum_{l=0}^{n-1} (4\eta^2)^l (\mathrm{D} + 4q\mathrm{D}^{-1}q)T\tilde{\Psi}^{n-l-1}\varphi_x + (4\eta^2)^n\varphi_x \\
&\qquad - \cos 2\varphi \sum_{l=1}^{n} 2^{2l-1}\eta^{2l-1}T\tilde{\Psi}^{n-l}\varphi_x + \sin 2\varphi \sum_{l=1}^{n} 2^{2l}\eta^{2l-1}\mathrm{D}^{-1}qT\tilde{\Psi}^{n-l}\varphi_x \\
&\quad = \sum_{l=1}^{n} 2^{2l-2}\eta^{2l-2}(\mathrm{D} + 4q\mathrm{D}^{-1}q - 2\eta\cos 2\varphi + 4\eta \sin 2\varphi\mathrm{D}^{-1}q) \\
&\qquad \times T\tilde{\Psi}^{n-l}\varphi_x + (4\eta^2)^n\varphi_x \\
&\quad = \sum_{l=1}^{n} 2^{2l-2}\eta^{2l-2}(\tilde{\Psi} - 4\eta^2)\tilde{\Psi}^{n-l}\varphi_x + 2^{2n}\eta^{2n}\varphi_x \\
&\quad = \tilde{\Psi}^n\varphi_x
\end{aligned}
$$

where $T = T(\varphi, \eta)$, $\Psi = \Psi(q)$, $\tilde{\Psi} = \tilde{\Psi}(\varphi, \eta)$. □

Lemma 2.2. If φ satisfies (2.7*b*), then q defined by (2.7*a*) satisfies the nth-order MKdV equation (2.1).

Proof. From (2.7), (2.8) and lemma 2.1 we have

$$
\begin{aligned}
q_t &= -(\mathrm{D} + 2\eta\cos 2\varphi)\varphi_t \\
&= -T\varphi_t \\
&= -T\tilde{\Psi}^n\varphi_x \\
&= -\Psi^n T\varphi_x \\
&= \Psi^n q_x
\end{aligned}
$$

which proves the lemma. □

Now we are prepared to construct a Bäcklund transformation for equation (2.1).

Theorem 2.1. Let $q(x, t)$ be a solution of equation (2.1), and V satisfy equation (2.2), and η_0 be any constant such that Re $\eta_0 < 0$ then

$$\bar{q} = q + 2(\varphi(x, t, \eta_0))_x = q + 2\varphi_{0x} = q + 2(\tan^{-1}\Gamma_0)_x \tag{2.9}$$

also satisfies equation (2.1).

Proof. From our assumption we know that (φ_0, η_0) satisfies equation (2.7*b*), and φ_0 has the required boundary condition. Since $\tilde{\Psi}(\varphi, \eta)$ is invariant under the transformation $(\varphi, \eta) - (-\varphi, -\eta)$ we know from lemma 2.1 that $(-\varphi_0, -\eta_0)$ also satisfies equation (2.7*b*). Define $\bar{q} = -(-\varphi_0)_x - (-\eta_0)\sin 2(-\varphi_0)$, then from lemma 2.2 we know that $\bar{q}$ also satisfies equation (2.1), which proves the theorem. □

Theorem 2.2. If $q(x, t)$ is a solution of equation (2.1), V satisfies equation (2.2), η_0, η are any constants such that Re $\eta_0 < 0$, Re $\eta < 0$, then $\bar{\varphi}$ defined by

$$\bar{\varphi} = \bar{\varphi}(\eta, \eta_0, x, t) = \tan^{-1}\left(\frac{-\eta\sin\varphi + \eta_0\sin(2\varphi_0 - \varphi)}{\eta\cos\varphi - \eta_0\cos(2\varphi_0 - \varphi)}\right)$$

is a solution of equation (2.7*a*) and (2.7*b*) with q replaced by

$$\bar{q} = q + 2\varphi(x, t, \eta_0)_x = q + 2\varphi_{0x}$$

i.e.

$$\bar{\varphi}(\eta, \eta_0, x, t)_t = \tilde{\Psi}^n(\bar{\varphi}, \eta)\bar{\varphi}_x \qquad \bar{\varphi}_x = -\bar{q} - \eta\sin 2\bar{\varphi}.$$

The proof of theorem 2.2 is given in appendix 1. From this theorem and lemma 2.2, we know that for any constant η, such that Re $\eta_1 < 0$ the $\bar{\bar{q}}$ defined by $\bar{\bar{q}} = \bar{q} + 2\bar{\varphi}(\eta_1, \eta_0, x, t)_x$ is also a solution of equation (2.1). So theorem 2.2 makes the further process of the Bäcklund transformation (2.9) an algebraic calculation.

3. Bäcklund transformations for the non-isospectral MKdV hierarchy

Consider the non-isospectral MKdV hierarchy

$$q_t = \Psi(q)^n(xq_x + q) \qquad n = 0, 1, 2, \ldots. \tag{3.1}$$

Equation (3.1) has the following Lax pair [13]:

$$V_x = MV \qquad V_t = N_nV \tag{3.2}$$

where

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad M = \begin{pmatrix} \eta & q \\ -q & -q \end{pmatrix} \qquad N_n = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix}$$

and the spectral parameter η satisfies $\eta_t = \eta(4\eta^2)^n$ (non-isospectral), A_n, B_n, C_n are defined by

$$B_n+C_n=\sum_{l=1}^{n}2^{2l}\eta^{2l-1}\Psi^{n-l}(xq)_x$$

$$=-\sum_{l=1}^{n}2^{2l}\eta^{2l-1}\Psi^{n-l}[(\mathrm{D}+2\eta\cos 2\varphi)(x\varphi_x)+\eta\sin 2\varphi]$$

$$=-\sum_{l=1}^{n}2^{2l}\eta^{2l-1}(\mathrm{D}+2\eta\cos 2\varphi)\tilde{\Psi}^{n-l}(x\varphi_x)$$

$$-\sum_{l=1}^{n}2^{2l}\eta^{2l-1}\Psi^{n-l}\eta\sin 2\varphi \qquad n=1,2,\ldots \tag{3.3}$$

$$B_0+C_0=0$$

$$B_n-C_n=\sum_{l=0}^{n-1}2^{2l+1}\eta^{2l}(\mathrm{D}+4q\mathrm{D}^{-1}q)\Psi^{n-l-1}(xq)x+2(4\eta^2)^n xq$$

$$=-\sum_{l=0}^{n-1}2^{2l+1}\eta^{2l}(\mathrm{D}+4q\mathrm{D}^{-1}q)(\mathrm{D}+2\eta\cos 2\varphi)\tilde{\Psi}^{n-l-1}(x\varphi_x)$$

$$-2(4\eta^2)^n x\varphi_x-\sum_{l=0}^{n-1}2^{2l+1}\eta^{2l}(\mathrm{D}+4q\mathrm{D}^{-1}q)\Psi^{n-l-1}\eta\sin 2\varphi$$

$$-2\eta(4\eta^2)^n x\sin 2\varphi \qquad n=0,1,2,\ldots \tag{3.4}$$

$$A_n=\mathrm{D}^{-1}q(B_n+C_n)+2(4\eta^2)^n x$$

$$=-\sum_{l=1}^{n}2^{2l}\eta^{2l-1}\mathrm{D}^{-1}q\Psi^{n-l}[(\mathrm{D}+2\eta\cos 2\varphi)(x\varphi_x)+\eta\sin 2\varphi]+\eta(4\eta^2)^n x$$

$$=-\sum_{l=1}^{n}2^{2l}\eta^{2l-1}\mathrm{D}^{-1}q(\mathrm{D}+2\eta\cos 2\varphi)\tilde{\Psi}^{n-l}(x\varphi_x)$$

$$-\sum_{l=1}^{n}2^{2l}\eta^{2l-1}\mathrm{D}^{-1}q\Psi^{n-l}\eta\sin 2\varphi+\eta(4\eta^2)^n x \qquad n=0,1,2,\ldots. \tag{3.5}$$

We define $\varphi=\tan^{-1}(v_2/v_1)$, then as in section 2 we have

$$\varphi_x=-q-\eta\sin 2\varphi \tag{3.6a}$$

$$\varphi_t=\tfrac{1}{2}(C_n-B_n)+\tfrac{1}{2}(C_n+B_n)\cos 2\varphi-A_n\sin 2\varphi. \tag{3.6b}$$

Lemma 3.1. If q and φ are related by (3.6a), then

$$\tfrac{1}{2}(C_n-B_n)+\tfrac{1}{2}(C_n+B_n)\cos 2\varphi-A_n\sin 2\varphi$$

$$=\sum_{l=1}^{n}(2\eta)^{2l-2}\tilde{\Psi}^{n-l}(4\eta^2\varphi_x\mathrm{D}^{-1}\sin^2 2\varphi-2\eta^2\sin 2\varphi\cos 2\varphi+2\eta\varphi_x)$$

$$+\tilde{\Psi}^n(x\varphi_x).$$

The proof is given in appendix 2. Now define

$$E(\varphi,\eta)=4\eta^2\varphi_x\mathrm{D}^{-1}\sin^2 2\varphi-2\eta^2\sin 2\varphi\cos 2\varphi+2\eta\varphi_x$$

then if q and φ are related by equation (3.6a), we have

$$\Psi(q)\eta\sin 2\varphi=T(\varphi,\eta)E(\varphi,\eta)+4\eta^3\sin 2\varphi. \tag{3.7}$$

Lemma 3.2. If q and φ are related by equation (3.6a), then

$$q_t-\Psi(q)^n(xq)_x=-T(\varphi,\eta)\left(\varphi_t-\tilde{\Psi}(\varphi,\eta)^n(x\varphi_x)-\sum_{l=1}^{n}(2\eta)^{2l-2}\tilde{\Psi}(\varphi,\eta)^{n-l}E(\varphi,\eta)\right).$$

Proof. By using identity (2.8) and (3.6a), we have

$$\begin{aligned}
q_t-\Psi(q)^n(xq)_x
&=-T(\varphi,\eta)\varphi_t-\eta(4\eta^2)^n\sin 2\varphi+\Psi(q)^nT(\varphi,\eta)(x\varphi_x)+\Psi(q)^n\eta\sin 2\varphi\\
&=-T(\varphi,\eta)\varphi_t+T\tilde{\Psi}(\varphi,\eta)^n(x\varphi_x)\\
&\quad+\sum_{l=1}^{n}2^{2l-2}\eta^{2l-2}\Psi(q)^{n-l}[\Psi(q)\eta\sin 2\varphi-4\eta^3\sin 2\varphi]\\
&=-T(\varphi,\eta)\varphi_t+T\tilde{\Psi}(\varphi,\eta)^n(x\varphi_x)+\sum_{l=1}^{n}2^{2l-2}\eta^{2l-2}\Psi(q)^{n-l}T(\varphi,\eta)E(\varphi,\eta)\\
&=-T(\varphi,\eta)\left(\varphi_t-\tilde{\Psi}(\varphi,\eta)^n(x\varphi_x)-\sum_{l=1}^{n}2^{2l-2}\eta^{2l-2}\tilde{\Psi}(\varphi,\eta)^{n-l}E(\varphi,\eta)\right)
\end{aligned}$$

above, we also used (3.7). The lemma is proved. □

Theorem 3.1. If q satisfies equation (3.1), V is a solution of equation (3.2), η_0 is any constant independent of x and satisfies the conditions $\eta_{0t}=\eta_0(4\eta_0^2)^n$, Re $\eta_0<0$, then

$$\bar{q}=q+2\varphi(x,t,\eta_0)_x=q+2\varphi_{0x} \tag{3.8}$$

satisfies the following equation

$$\bar{q}_t-\Psi(\bar{q})^n(x\bar{q})_x-\sum_{l=1}^{n}2(2\eta_0)^{2l-1}\Psi(\bar{q})^{n-l}\bar{q}_x=0. \tag{3.9}$$

Proof. Denote $\bar{\varphi}=-\varphi_0$, $\bar{\eta}=-\eta_0$, then by using the definitions of $\bar{q}$ and $\tilde{\Psi}$, lemma 3.1, lemma 3.2 and identity (2.8) we have

$$\bar{q}=-\bar{\varphi}_x-\bar{\eta}\sin 2\bar{\varphi}$$

$$\begin{aligned}
\bar{q}_t-\Psi(\bar{q})^n(x\bar{q})_x
&=-T(\bar{\varphi},\bar{\eta})\left(\bar{\varphi}_t-\tilde{\Psi}(\bar{\varphi},\bar{\eta})^n(x\bar{\varphi}_x)-\sum_{l=1}^{n}(2\bar{\eta})^{2l-2}\tilde{\Psi}(\bar{\varphi},\bar{\eta})^{n-l}E(\bar{\varphi},\bar{\eta})\right)\\
&=-T(\bar{\varphi},\bar{\eta})\Bigg(-\varphi_{0t}+\tilde{\Psi}(\varphi_0,\eta_0)^n(x\varphi_{0x})+\sum_{l=1}^{n}(2\eta_0)^{2l-2}\tilde{\Psi}(\varphi_0,\eta_0)^{n-l}E(\varphi_0,\eta_0)\\
&\quad-\sum_{l=1}^{n}(2\eta_0)^{2l-2}\tilde{\Psi}(\varphi_0,\eta_0)^{n-l}4\eta_0\varphi_{0x}\Bigg)\\
&=-T(\bar{\varphi},\bar{\eta})\left(-\sum_{l=1}^{n}(2\bar{\eta})^{2l-2}\tilde{\Psi}(\bar{\varphi},\bar{\eta})^{n-l}4\bar{\eta}\bar{\varphi}_x\right)\\
&=\sum_{l=1}^{n}2(2\eta_0)^{2l-1}\Psi(\bar{q})^{n-l}\bar{q}_x
\end{aligned}$$

thus the theorem is proved. □

Theorem 3.2. Let q, V and η_0 satisfy the assumptions of theorem 3.1, and η satisfy same conditions as η_0 does. Define

$$\varphi^{(1)}=\tan^{-1}\left(\frac{-\eta\sin\varphi+\eta_0\sin(2\varphi_0-\varphi)}{\eta\cos\varphi-\eta_0\cos(2\varphi_0-\varphi)}\right) \tag{3.10}$$

then $\varphi^{(1)}$ satisfies the following equations:

$$\varphi_x^{(1)}=-\bar{q}-\eta\sin 2\varphi^{(1)} \tag{3.11a}$$

$$\varphi_t^{(1)}=\tilde{\Psi}(\varphi^{(1)},\eta)(x\varphi_x^{(1)})+\sum_{l=1}^{n}(2\eta)^{2l-2}\tilde{\Psi}(\varphi^{(1)},\eta)^{n-l}E(\varphi^{(1)},\eta)$$
$$+\sum_{l=1}^{n}2(2\eta_0)^{2l-1}\tilde{\Psi}(\varphi^{(1)},\eta)^{n-l}\varphi_x^{(1)}. \tag{3.11b}$$

The proof of (3.11a) is same as the proof given in appendix 1. From the identity (2.8) and (3.11a) we have

$$0=\bar{q}_t-\Psi(\bar{q})^n(x\bar{q})_x-\sum_{l=1}^{n}2(2\eta_0)^{2l-1}\Psi(\bar{q})^{n-l}\bar{q}_x$$
$$=-T(\varphi^{(1)},\eta)\bigg(\varphi_t^{(1)}-\tilde{\Psi}(\varphi^{(1)},\eta)(x\varphi_z^{(1)})$$
$$-\sum_{l=1}^{n}(2\eta)^{2l-2}\tilde{\Psi}(\varphi^{(1)},\eta)^{n-l}E(\varphi^{(1)},\eta)-\sum_{l=1}^{n}2(2\eta_0)^{2l-1}\tilde{\Psi}(\varphi^{(1)},\eta)^{n-l}\varphi_x^{(1)}\bigg).$$

The derivation of the above identity is similar to that of lemma 3.2. Since $E(-\varphi^{(1)},\eta)=-E(\varphi^{(1)},\eta)$, we can prove (3.11$b$) by following the argument given in appendix 1.

Theorem 3.2 makes the further process of the Bäcklund transformation (3.8) an algebraic calculation. To state more explicitly, we denote $\varphi^{(1)}$ defined by (3.10) by $\varphi^{(1)}(x,t,\eta_0,\eta)$. Let

$$\varphi^{(0)}(x,t,\eta)=\varphi(x,t,\eta)$$

$$\varphi^{(k)}(x,t,\eta_0,\eta_1,\ldots,\eta_{k-1},\eta)=\tan^{-1}\left(\frac{-\eta\sin\varphi^{(k-1)}+\eta_{k-1}\sin(2\varphi_{k-1}^{(k-1)}-\varphi^{(k-1)})}{\eta\cos\varphi^{(k-1)}-\eta_{k-1}\cos(2\varphi_{k-1}^{(k-1)}-\varphi^{(k-1)})}\right)$$

where

$$\varphi_k^{(k)}=\varphi^{(k)}(x,t,\eta_0,\eta_1,\ldots,\eta_{k-1},\eta_k)$$

and $\eta_0,\eta_1,\ldots,\eta_k$ are defined as η_0 given in theorem 3.1. Then by using theorem 3.2 and following the similar argument as the one given in the proof of theorem 3.1, we know that $q^{(k)}$ defined by

$$q^{(0)}=q$$
$$q^{(k)}=q^{(k-1)}+2\varphi_{k-1x}^{(k-1)}\qquad k=1,2,\ldots \tag{3.12}$$

satisfies the following equation:

$$q_t^{(k)}-\Psi(q^{(k)})^n(xq^{(k)})_x-\sum_{l=1}^{n}\sum_{j=0}^{k-1}2(2\eta_j)^{2l-1}\Psi(q^{(k)})^{n-l}q_x^{(k)}=0. \tag{3.13}$$

Unlike the isospectral case, for the non-isospectral case the Bäcklund transformation (3.12) is not an auto-Bäcklund transformation. Since the Lax pair of equation (3.13) can be obtained by a linear combination of the Lax pairs (2.2) and (3.2), from the derivation of (3.2) we know that we can also start with the equation (3.13) instead of the equation (3.1) to obtain a Bäcklund transformation similar to (3.12). For example, we can start with the solution $\tilde{q}(x, t)$ of the following equation:

$$q_t - \Psi(q)^n (xq)_x + \sum_{l=1}^{n} \sum_{j=0}^{k-1} 2(2\eta_j)^{2l-1} \Psi(q)^{n-l} q_x = 0. \tag{3.14}$$

Firstly define $\tilde{\varphi}^{(0)}(x, t, \eta)$ by using the Lax pair of equation (3.14) as we define φ at the beginning of this section, then from $\tilde{\varphi}^{(0)}$ we define $\tilde{\varphi}^{(k)}$ as we define $\varphi^{(k)}$. Let

$$\tilde{q}^{(0)} = \tilde{q} \qquad \tilde{q}^{(k)} = \tilde{q}^{(k-1)} + 2\tilde{\varphi}^{(k-1)}_{k-1x} \qquad k = 1, 2, \ldots$$

then $\tilde{q}^{(k)}$ satisfies the equation (3.1).

Remark. The equation $\varphi_t = \tilde{\Psi}\varphi_x = \varphi_{xxx} + 2\varphi_x^3 + 6\eta^2 \varphi_x \sin^2 2\varphi$ is called the CDF (Calogero–Degasperies–Fordy) equation [14]. In this paper we have obtained its hereditary symmetry $\tilde{\Psi}$, and the CDF hierarchy. So we can, further, get two infinite sets of symmetries and their Lie algebraic structure for this hierarchy of equations. We can also study the symmetries and the related Lie algebraic structure for the hierarchy of equations given by (3.6*b*).

Acknowledgment

This project is supported by the National Natural Science Foundation of China.

Appendix 1. Proof of theorem 2.2

Firstly, we define

$$\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} = \begin{pmatrix} \eta - \eta_0 \cos 2\varphi_0 & -\eta_0 \sin 2\varphi_0 \\ \eta_0 \sin 2\varphi_0 & -\eta - \eta_0 \cos 2\varphi_0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

then from equation (2.2) we have

$$\begin{aligned} \bar{v}_{1x} - (\eta\bar{v}_1 + \bar{q}\bar{v}_2) &= 2\eta_0\varphi_{0x} \sin 2\varphi_0 v_1 + (\eta - \eta_0 \cos 2\varphi_0)(\eta v_1 + q v_2) - 2\eta_0 \varphi_{0x} \cos 2\varphi_0 v_2 \\ &\quad - \eta_0 \sin 2\varphi_0(-q v_1 - \eta v_2) - [\eta(\eta - \eta_0 \cos 2\varphi_0) v_1 - \eta\eta_0 \sin 2\varphi_0 v_2 \\ &\quad + (q + 2\varphi_{0x})\eta_0 \sin 2\varphi_0 v_1 - (q + 2\varphi_{0x})(\eta + \eta_0 \cos 2\varphi_0) v_2] \\ &= 2\eta_0\eta \sin 2\varphi_0 v_2 + 2\eta q v_2 + 2\eta\varphi_{0x} v_2 \\ &= 0. \end{aligned}$$

Similarly we have

$$\bar{v}_{2x} = -\eta\bar{v}_2 - q\bar{v}_1 .$$

So

$$\begin{pmatrix} \bar{v}_{1x} \\ \bar{v}_{2x} \end{pmatrix} = \begin{pmatrix} \eta & \bar{q} \\ -\bar{q} & -\eta \end{pmatrix} \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}.$$

Define $\bar{\varphi}$ as follows:

$$\begin{aligned} \bar{\varphi} &= \tan^{-1} \bar{\Gamma} \\ &= \tan^{-1}\left(\frac{\bar{v}_2}{\bar{v}_1}\right) \\ &= \tan^{-1}\left(\frac{\eta_0 \sin 2\varphi_0 - (\eta + \eta_0 \cos 2\varphi_0) \tan \varphi}{(\eta - \eta_0 \cos 2\varphi_0) - \eta_0 \sin 2\varphi_0 \tan \varphi}\right) \\ &= \tan^{-1}\left(\frac{\eta_0 \sin(2\varphi_0 - \varphi) - \eta \sin \varphi}{\eta \cos \varphi - \eta_0 \cos(2\varphi_0 - \varphi)}\right). \end{aligned}$$

Then by following the derivation of (2.7) we have

$$\bar{q} = -\bar{\varphi}_x - \eta \sin 2\bar{\varphi}.$$

From (2.8) and the above identity we get

$$(\bar{q}_t - \Psi(\bar{q})^n \bar{q}_x) = -(\mathrm{D} + 2\eta \cos 2\bar{\varphi})(\bar{\varphi}_t - \tilde{\Psi}(\bar{\varphi}, \eta)^n \bar{\varphi}_x).$$

However, theorem 2.1 insures that $\bar{q}$ satisfies equation (2.1), so

$$\bar{\varphi}_t = \tilde{\Psi}(\bar{\varphi}, \eta)^n \bar{\varphi}_x + f(t, \eta, \eta_0) \exp\left(-2\eta \int_{-x}^{x} (\cos 2\bar{\varphi} - 1)\, \mathrm{d}x' - 2\eta x\right).$$

From the definition of $\bar{\varphi}$ we can write $\bar{\varphi}_t$ as

$$F\left(\varphi_0, \frac{\partial \varphi_0}{\partial x}, \ldots, \frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}, \varphi, \ldots, \frac{\partial^{2n+1} \varphi}{\partial x^{2n+1}}\right)$$

by using the fact that $\tilde{\Psi}(-\varphi, \eta) = \tilde{\Psi}(\varphi, \eta)$ we have

$$F\left(-\varphi_0, \ldots, -\frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}, -\varphi, \ldots, -\frac{\partial^{2n+1} \varphi}{\partial x^{2n+1}}\right) = F\left(\varphi_0, \ldots, \frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}, \varphi, \ldots, \frac{\partial^{2n+1} \varphi}{\partial x^{2n+1}}\right).$$

On the other hand, we can denote $\tilde{\Psi}(\bar{\varphi}, \eta)^n \bar{\varphi}_x$ by

$$H\left(\varphi_0, \ldots, \frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}, \varphi, \ldots, \frac{\partial^{2n+1} \varphi}{\partial x^{2n+1}}\right)$$

and we have

$$H\left(-\varphi_0, \ldots, -\frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}, -\varphi, \ldots, -\frac{\partial^{2n+1} \varphi}{\partial x^{2n+1}}\right) = -H\left(\varphi_0, \ldots, \frac{\partial^{2n+1} \varphi_0}{\partial x^{2n+1}}, \varphi, \ldots, \frac{\partial^{2n+1} \varphi}{\partial x^{2n+1}}\right).$$

So we get $f(t, \eta, \eta_0) = 0$. Thus the theorem is proved. □

Appendix 2. Proof of lemma 3.1

From (3.3)-(3.5) and (3.6a) we get

$$\tfrac{1}{2}(C_n - B_n) + \tfrac{1}{2}(C_n + B_n)\cos 2\varphi - A_n \sin 2\varphi$$

$$= \sum_{l=0}^{n-1} 2^{2l}\eta^{2l}(\mathrm{D}+4q\mathrm{D}^{-1}q)T\tilde{\Psi}^{n-l-1}(x\varphi_x) + 2^{2n}\eta^{2n}x\varphi_x$$

$$+\sum_{l=0}^{n-1} 2^{2l}\eta^{2l}(\mathrm{D}+4q\mathrm{D}^{-1}q)\Psi^{n-l-1}\eta\sin 2\varphi + 2^{2n}\eta^{2n+1}x\sin 2\varphi$$

$$-\cos 2\varphi \sum_{l=1}^{n} 2^{2l-1}\eta^{2l-1}T\tilde{\Psi}^{n-l}x\varphi_x - \cos 2\varphi \sum_{l=1}^{n} 2^{2l-1}\eta^{2l-1}\Psi^{n-l}\eta\sin 2\varphi$$

$$+\sin 2\varphi \sum_{l=1}^{n} 2^{2l}\eta^{2l-1}\mathrm{D}^{-1}qT\tilde{\Psi}^{n-l}(x\varphi_x)$$

$$+\sin 2\varphi \sum_{l=1}^{n} 2^{2l}\eta^{2l-1}\mathrm{D}^{-1}q\Psi^{n-l}\eta\sin 2\varphi - 2^{2n}\eta^{2n+1}x\sin 2\varphi$$

$$= \sum_{l=1}^{k} 2^{2l-2}\eta^{2l-2}(\mathrm{D}+4q\mathrm{D}^{-1}q - 2\eta\cos 2\varphi + 4\eta\sin 2\varphi\mathrm{D}^{-1}q)$$

$$\times T\tilde{\Psi}^{n-l}(x\varphi_x) + 2^{2n}\eta^{2n}x\varphi_x$$

$$+\sum_{l=1}^{n} 2^{2l-2}\eta^{2l-2}(\mathrm{D}+4q\mathrm{D}^{-1}q - 2\eta\cos 2\varphi + 4\eta\sin 2\varphi\mathrm{D}^{-1}q)\Psi^{n-l}\eta\sin 2\varphi$$

$$= \tilde{\Psi}^{n}(x\varphi_x) + S$$

where

$$S = \sum_{l=1}^{n} 2^{2l-2}\eta^{2l-2}(\mathrm{D}+4q\mathrm{D}^{-1}q - 2\eta\cos 2\varphi + 4\eta\sin 2\varphi\mathrm{D}^{-1}q)\Psi^{n-l}\eta\sin 2\varphi.$$

By using the identity (3.7) we have

$$S = \sum_{l=1}^{n-1} 2^{2l-2}\eta^{2l-2}(\mathrm{D}+4q\mathrm{D}^{-1}q - 2\eta\cos 2\varphi + 4\eta\sin 2\varphi\mathrm{D}^{-1}q)\Psi^{n-l-1}(TE + 4\eta^3\sin 2\varphi)$$

$$+2^{2n-2}\eta^{2n-2}E$$

$$= \sum_{l=1}^{n-1} 2^{2l-2}\eta^{2l-2}(\mathrm{D}+4q\mathrm{D}^{-1}q - 2\eta\cos\varphi + 4\eta\sin 2\varphi\mathrm{D}^{-1}q)T\tilde{\Psi}^{n-l-1}E$$

$$+2^{2n-2}\eta^{2n-2}E + 4\eta^2\sum_{l=1}^{n-1} 2^{2l-2}\eta^{2l-2}$$

$$\times(\mathrm{D}+4q\mathrm{D}^{-1}q - 2\eta\cos 2\varphi + 4\eta\sin 2\varphi\mathrm{D}^{-1}q)\Psi^{n-l-1}\eta\sin 2\varphi$$

$$= \sum_{l=1}^{n-1} 2^{2l-2}\eta^{2l-2}(\tilde{\Psi} - 4\eta^2)\tilde{\Psi}^{n-l-1}E + 2^{2n-2}\eta^{2n-2}E + 4\eta^2\sum_{l=1}^{n-1} 2^{2l-2}\eta^{2l-2}$$

$$\times(\mathrm{D}+4q\mathrm{D}^{-1}q - 2\eta\cos 2\varphi + 4\eta\sin 2\varphi\mathrm{D}^{-1}q)\Psi^{n-l-1}\eta\sin 2\varphi$$

$$= \sum_{l=1}^{n} 2^{2l-2}\eta^{2l-2}\tilde{\Psi}^{n-l}E.$$

Thus the lemma is proved. □

References

[1] Miura R M *et al* 1974 *Bäcklund Transformations* (*Lecture Notes in Math.* **515**) (Berlin: Springer)
[2] Rogers C and Shadwick W R 1982 *Bäcklund Transformation Applications* (New York: Academic)
[3] Li Y S 1989 Gauge transformation, Bäcklund transformation and nonlinear superposition laws *Adv. Math.* **18** 356-72
[4] Wahlquist H D and Estabrook F 1973 *J. Math. Phys.* **10** 1-7
[5] Wadati M, Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 417-36
[6] Tian C 1985 *Acta Math. Appl. Sinica* (English series) **2** 89-94
[7] Gu C H and Hu H S 1986 *Lett. Math. Phys.* **11** 325
[8] Gu C H 1986 *Lett. Math. Phys.* **12** 31
[9] Gu C H and Zhou Z X 1987 *Lett. Math. Phys.* **13** 179
[10] Li Y S, Gu X S and Zou M R 1987 *Acta Math. Sinica* (New series) **3** 143
[11] Gu X S 1987 *Ann. Diff. Eqs.* **3** 13
[12] Tian C and Zhang Y J 1989 Bäcklund transformations for the isospectral and non-isospectral Kdv hierarchies *Nonlinear Physics* ed C C Gu *et al* (Berlin: Springer) in press
[13] Li Y S 1982 *Sci. Sinica* A **25** 385
[14] Abellanas L and Galindo A 1985 *Phys. Lett.* **108A** 123